ÉRETTSÉGI VIZSGA • 2021. május 4.

MATEMATIKA ANGOL NYELVEN

EMELT SZINTŰ ÍRÁSBELI VIZSGA

JAVÍTÁSI-ÉRTÉKELÉSI ÚTMUTATÓ

EMBERI ERŐFORRÁSOK MINISZTÉRIUMA

Instructions to examiners

Formal requirements:

1. Mark the paper **legibly, in ink, different in colour** from that used by the candidate.
2. The first of the rectangles next to each problem shows the maximum attainable score on that problem. The **score** given by the examiner is to be entered **into the rectangle** next to this.
3. **If the solution is perfect**, enter maximum score and, with a checkmark, indicate that you have seen each item leading to the solution and consider them correct.
4. If the solution is incomplete or incorrect, please **mark the error** and also indicate the individual **partial scores**. It is also acceptable to indicate the points lost by the candidate if it makes grading the paper easier. After correcting the paper, it must be clear about every part of the solution whether that part is correct, incorrect or unnecessary.
5. Please, **use the following symbols** when correcting the paper:
   - correct calculation: *checkmark*
   - principal error: *double underline*
   - calculation error or other, not principal, error: *single underline*
   - correct calculation with erroneous initial data: *dashed checkmark or crossed checkmark*
   - incomplete reasoning, incomplete list, or other missing part: *missing part symbol*
   - unintelligible part: *question mark* and/or *wave*
6. Do not assess anything written **in pencil**, except for diagrams

Assessment of content:

1. The answer key may contain more than one solution for some of the problems. If the **solution given by the candidate is different**, allocate the points by identifying parts of the solution equivalent to those given in the answer key.
2. Subtotals may be **further divided, unless stated otherwise in the answer key**. However, scores awarded must always be whole numbers.
3. If there is a **calculation error** or inaccuracy in the solution, take points off only for the part where the error occurs. If the reasoning remains correct and the error is carried forward while the nature of the problem remains unchanged, points for the rest of the solution must be awarded.
4. **In case of a principal error**, no points should be awarded at all for that section of the solution, not even for steps that are formally correct. (These logical sections of the solutions are separated by double lines in the answer key.) However, if the erroneous information obtained through principal error is carried forward to the next section or the next part of the problem, and it is used there correctly, the maximum score is due for that part, provided that the error has not changed the nature of the task to be completed.
5. Where the answer key shows a **unit** or a **remark** in brackets, the solution should be considered complete without that unit or remark as well.

6. If there are more than one different approach to a problem, **assess only the one indicated by the candidate**. Please, mark clearly which attempt was assessed.
7. **Do not give extra points** (i.e. more than the score due for the problem or part of problem).
8. The score given for the solution of a problem, or part of a problem, **may never be negative**.
9. **Do not take points off** for steps or calculations that contain errors but are not actually used by the candidate in the solution of the problem.
10. **The use of calculators** in the reasoning behind a particular solution **may be accepted without further mathematical explanation in case of the following operations**: addition, subtraction, multiplication, division, calculating powers and roots, $n!$, $\binom{n}{k}$, replacing the tables found in the 4-digit Data Booklet (sin, cos, tan, log, and their inverse functions), approximate values of the numbers π and e, finding the solutions of the standard quadratic equation. No further explanation is needed when the calculator is used to find the mean and the standard deviation, as long as the text of the question does not explicitly require the candidate to show detailed work. **In any other cases, results obtained through the use of a calculator are considered as unexplained and points for such results will not be awarded.**
11. Using **diagrams** to justify a solution (e.g. reading data off diagrams by way of direct measurement) is unacceptable.
12. **Probabilities** may also be given in percentage form (unless stated otherwise in the text of the problem).
13. If the text of the problem does not instruct the candidate to round their solution in a particular way, any solution **rounded reasonably and correctly** is acceptable even if it is different from the one given in the answer key.
14. **Assess only four out of the five problems in part II of this paper**. The candidate was requested to indicate in the appropriate square the number of the problem not to be assessed and counted towards their total score. Should there be a solution to that problem, it does not need to be marked. However, if the candidate did not make a selection and neither is their choice clear from the paper, assume automatically that it is the last problem in the examination paper that is not to be assessed.

I.

1. a)		
Among six consecutive natural numbers there must be two that are divisible by 3,	1 point	
and at least one that is divisible by 5,	1 point	
and so the product of the six numbers will be divisible by $3 \cdot 3 \cdot 5 = 45$.	1 point	
Total:	**3 points**	

1. b)		
The statement is not true.	1 point	
A counterexample: $11 \cdot 13 \cdot 15 \cdot 17 \cdot 19$ (= 692 835).	2 points	*If the middle one of five consecutive odd natural numbers is divisible by* 3 *but is not divisible by* 9 *then the product will not be divisible by* 9 *and therefore will not be divisible by* 45 *either.*
Total:	**3 points**	

<table>
<tr><th colspan="3">1. c)</th></tr>
<tr><td>(As $a + b + c = 37$,) if $a = 7$ then ($b + c = 30$) the possible values of b and c are: (9; 21), (11; 19) and (13; 17). This is a total of 3 different solutions.</td><td>1 point</td><td rowspan="3"><i>(As</i> 7 + 9 = 16<i>, the maximum value of c is</i> 21.
<i>Also,</i> $9 + 11 + 13 < 37$<i>, so the minimum value of c is</i> 15.<i>)</i>
<i>Possible solutions:</i>
<table>
<tr><th><i>c</i></th><th><i>b</i></th><th><i>a</i></th></tr>
<tr><td>21</td><td>9</td><td>7</td></tr>
<tr><td>19</td><td>11</td><td>7</td></tr>
<tr><td>17</td><td>13</td><td>7</td></tr>
<tr><td>17</td><td>11</td><td>9</td></tr>
<tr><td>15</td><td>13</td><td>9</td></tr>
</table></td></tr>
<tr><td>If $a = 9$ then ($b + c = 28$) the possible values of b and c are: (11; 17) and (13; 15).
This is a further 2 different solutions.</td><td>1 point</td></tr>
<tr><td>If $a \geq 11$ then (since $a < b < c$) $b \geq 13$ and $c \geq 15$, so $a + b + c \geq 39$ that leads to a contradiction.</td><td>1 point</td></tr>
<tr><td>The values of a, b and c that satisfy the conditions will provide 5 different solutions.</td><td>1 point</td><td></td></tr>
<tr><td>Total:</td><td>4 points</td><td></td></tr>
</table>

Notes:

1. If the candidate lists the possible triples but the list contains incorrect items, missing items or items listed multiple times deduce 1 point for each incorrect (missing or multiple) items (but no more than 3 points altogether) from the 3 points allocated for this part of the problem.

2. Award a maximum of 2 points if the candidate ignores one of the three conditions given (parity, difference, order of numbers).

1. d)

<table>
<tr><td>
<table>
<tr><th>A</th><th>B</th><th>C</th><th>$(A \lor B) \to C$</th></tr>
<tr><td>t</td><td>t</td><td>t</td><td>t</td></tr>
<tr><td>t</td><td>t</td><td>f</td><td>f</td></tr>
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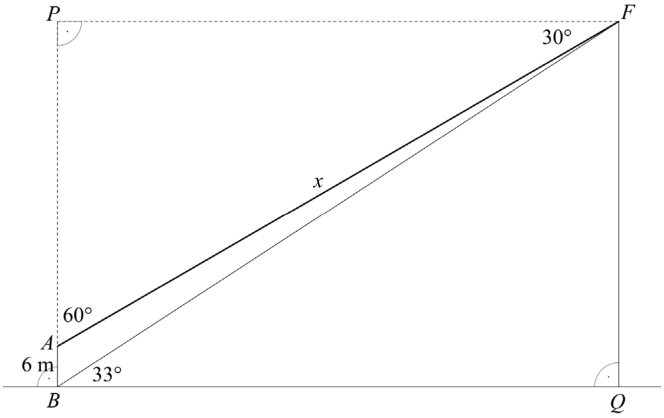
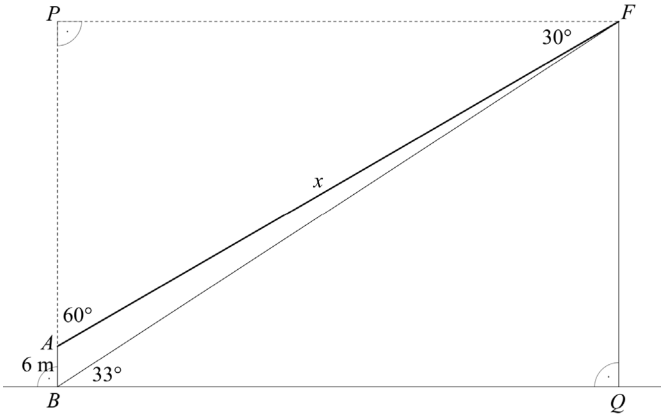
<tr><td>f</td><td>f</td><td>t</td><td>t</td></tr>
<tr><td>f</td><td>f</td><td>f</td><td>t</td></tr>
</table>
</td><td>3 points</td><td><i>Deduce 1 point for each missing or incorrect answer (however, the total score may not be negative).</i></td></tr>
<tr><td align="right">Total:</td><td>3 points</td><td></td></tr>
</table>

2. a) Solution 1

Sketch, based on the text:  A is the lower terminus, F is the upper terminus, B is the location of the lower terminus according to the original plans, the length of the track is $AF = x$.	2 points	*Correct diagram in accordance with the text, indicating relevant data.*
In triangle ABF: $ABF\angle = 57°$ and $BFA\angle = 3°$.	1 point	
Apply the Law of Sines: $\frac{x}{6} = \frac{\sin 57^\circ}{\sin 3^\circ}$.	1 point	
The length of the track is $x \left(= \frac{6 \cdot \sin 57^\circ}{\sin 3^\circ} \right) \approx 96$ m.	1 point	
In the right triangle APF: $AP = x \cdot \sin 30^\circ = \frac{x}{2}$,	1 point	
The elevation of the track is $AP \approx 48$ m.	1 point	
Total:	**7 points**	

2. a) **Solution 2**		
Sketch, based on the text:  *A* is the lower terminus, *F* is the upper terminus, *B* is the location of the lower terminus according to the original plans, the length of the track is $AF = x$.	2 points	*Correct diagram in accordance with the text, indicating relevant data.*
In the right triangle *APF* (which is half of a regular triangle): $AP = \frac{x}{2}$ and $PF = \frac{x\sqrt{3}}{2}$.	1 point	
In the right triangle *BPF: PBF*∠ = 57° and $BP = AP + 6$, and so $\tan 57^\circ = \frac{\frac{x\sqrt{3}}{2}}{\frac{x}{2}+6}$.	1 point	
$\left(\frac{x}{2}+6\right)\cdot\tan 57^\circ = \frac{x\sqrt{3}}{2}$ $(0.5x+6)\cdot 1.540 \approx 0.866x$ $0.096x \approx 9.24$	1 point	$x = \frac{12\cdot\tan 57^\circ}{\sqrt{3}-\tan 57^\circ}$
The length of the track is $x \approx 96$ m.	1 point	
The elevation of the track (half of the length) is 48 m.	1 point	
Total:	**7 points**	

Note: Deduce a total of 1 point if the candidate gives an answer without a unit.

2. b) **Solution 1**		
The funicular service carried an average of $\frac{670\,000}{340}$ (≈ 1971) passengers a day during the year of 1896.	1 point	
The daily operation time was 14 hours = 840 minutes. This allows about 84 rides a day.	1 point	*Accept 85 rides, too.*
The average number of passengers per ride is $\frac{\frac{670\,000}{340}}{84} \approx 23.5$.	1 point	
As the overall capacity of the two cars is 44 passengers,	1 point	
the average occupancy rate of the seats in 1896 was about $\left(\frac{23.5}{44}\cdot 100 \approx\right)$ 53 percent.	1 point	
Total:	**5 points**	

2. b) **Solution 2**		
The daily operation time was 14 hours = 840 minutes. This allows about 84 rides a day.	1 point	*Accept 85 rides, too.*
The overall capacity of the two cars is 44 passengers.	1 point	
If the cars had carried 44 passengers each ride (100% occupancy rate), a total of about $84 \cdot 44 \cdot 340 \approx$ 1 257 000 passengers would have been carried during the year.	2 points	
The average occupancy rate of the seats in 1896 was therefore about $\left(\frac{670\,000}{1\,257\,000}\cdot 100 \approx\right)$ 53 percent.	1 point	
Total:	**5 points**	

Note: Accept any reasonable and practical estimate and rounding.

3. a)

Let x be the number of girls attending the lecture. The number of boys is then $32 - x$ ($x \in \mathbf{N}$ and $x \geq 4$).	1 point	*Let y be the number of boys attending the lecture. The number of girls is then $32 - y$ ($y \in \mathbf{N}$ and $y \leq 28$).*
If 4 girls leave, the number of remaining students drops to 28, while the number of boys remains unchanged: $32 - x > 0.6 \cdot 28$ $(= 16.8)$, that is $x < 15.2$.	1 point	$y > 0.6 \cdot 28$ $(= 16.8)$, *that is* $y > 16.8$
If 6 girls join the 32 students in the hall, the number of attending students rises to 38, $x + 6$ of whom are girls: $x + 6 > 0.5 \cdot 38$ $(= 19)$, that is $x > 13$.	1 point	$38 - y > 0.5 \cdot 38$ $(= 19)$, *that is* $y < 19$
As $13 < x < 15.2$, the number of girls is either 14 or 15 and, consequently, the number of boys is 18 or 17.	2 points	
Check based on the original text. (E.g. Both 17 and 18 are greater than 60% of the 28 remaining students (16.8) after the 4 girls leave. Both 20 (=14 + 6) and 21 (=15 + 6) are greater than half of the 38 students (19) after the arrival of 6 more girls.)	1 point	
Total:	**6 points**	

3. b)

(Apply the binomial distribution.) One has to find the probability that there will be 3 or 4 boys among the four randomly selected students.	1 point	*This point is also due if the correct reasoning is reflected only by the solution.*
$P(3 \text{ boys}) = \binom{4}{3} \cdot 0.6^3 \cdot 0.4$ $(= 0.3456)$	1 point	
$P(4 \text{ boys}) = 0.6^4$ $(= 0.1296)$	1 point	
The final probability is the sum of the above, approximately 0.475.	1 point	
Total:	**4 points**	

Note: Award a maximum of 3 points if the candidate applies the hypergeometric distribution assuming an arbitrary number of university students, as long as it otherwise corresponds to the text of the original problem.

3. c)

(Estimate with the binomial distribution.) Assuming the proportion of active sportswomen is p ($0 \leq p \leq 1$), the probability that a randomly selected girl is an active sportswoman is the same p.	1 point	*This point is also due if the correct reasoning is reflected only by the solution.*
$p^3 = 0.008$,	1 point	
$p = \sqrt[3]{0.008} = 0.2$, i.e. one fifth of the girls are active sportswomen.	1 point	
Total:	**3 points**	

4. a)

From the equation $y = \frac{x^2}{4}$ of the parabola: $p = 2$.	1 point	
As the vertex of the parabola is the origin, $\frac{p}{2} = 1$ (and the parabola is an “upright” one), the focus is indeed $F(0; 1)$.	2 points	*Award these two points for a correct diagram, too.*
Total:	**3 points**	

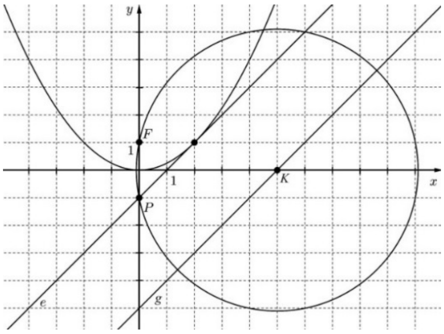
4. b)

The centre of the circle will be the intersection of the perpendicular bisector of segment *PF* and the line *g*.	1 point	*This point is also due if the correct reasoning is reflected only by the solution.*
The perpendicular bisector is the x-axis,	1 point	
the centre of the circle is therefore the point $K(5; 0)$.	1 point	
The radius of the circle: $(\sqrt{5^2+1^2} =) \sqrt{26}$.	1 point	
The equation of the circle: $(x-5)^2 + y^2 = 26$.	1 point	$x^2 + y^2 - 10x - 1 = 0$
Total:	**5 points**	

4. c) Solution 1

The equation of tangent *e* may also be given in the form $x - y = c$.	1 point	*This point is also due if the correct reasoning is reflected only by the solution.*
Line *e* is tangent to the parabola if and only if the equation system below has exactly one solution: $\left.\begin{array}{r} x^2 - 4y = 0 \\ x - y = c \end{array}\right\\}$	1 point	
From the second equation $y = x - c$. Substitute it into the first equation: $x^2 - 4x + 4c = 0$.	1 point	
(There is exactly one solution when the discriminant is 0) $16 - 16c = 0$, from where $c = 1$.	1 point	
The equation of the tangent line is: $x - y = 1$.	1 point	
Total:	**5 points**	

4. c) Solution 2		
The gradient of the tangent is equal to that of line g: 1.	1 point	
(The parabola $y = \frac{1}{4}x^2$ is actually the graph of the quadratic function $x \mapsto \frac{1}{4}x^2$.) The derivative of the function $x \mapsto \frac{1}{4}x^2$ is $x \mapsto \frac{1}{2}x$.	1 point	
(The first derivative gives the gradient of the tangent drawn to the function at the given point:) $\frac{1}{2}x = 1$.	1 point	
From this $x = 2$, i.e. the point of tangency is (2; 1).	1 point	
The equation of the tangent is $y - 1 = x - 2$.	1 point	$x - y = 1$ or $y = x - 1$
Total:	**5 points**	



II.

5. a)		
Let n be the number of years passed. In this case: $5 \cdot 0.88^n < 1.5$, that is $0.88^n < 0.3$, where n is a positive integer.	2 points	
The base 0.88 exponential function (as well as the base 0.88 logarithm function) is strictly monotone decreasing:	1 point	*Taking the base 10 logarithm of both sides (and considering the increasing nature of the base 10 logarithm function):* $\log 0.88^n < \log 0.3$, $n \cdot \log 0.88 < \log 0.3$.
$n > \log_{0.88} 0.3 \approx 9.42$.	1 point	$\log 0.88$ *is negative, so* $n > \frac{\log 0.3}{\log 0.88} \approx 9.42$.
The value of the car will drop below 1.5 million forints after 10 full years.	1 point	
Total:	**5 points**	

Note: Award full score if the candidate calculates the current value of the car each year (and this is clearly documented) or if the candidate gives the correct answer by solving an equation instead of an inequality.

5. b)		
Assuming that the value of the car changes by a factor of q each month, by the end of the 12th month it will change to q^{12} times what it was at the beginning of the year ($0 < q < 1$).	1 point	
$q^{12} = (1 - 0.12 =)\ 0.88$	1 point	
$q = \sqrt[12]{0.88} \approx 0.9894$ (this is 98.94%).	1 point	
So, the monthly depreciation of the value of the car is approximately (100 – 98.94 =) 1.06%, indeed.	1 point	
Total:	**4 points**	

Note: Award 3 points if the candidate proves that a monthly depreciation of 1.06% is equivalent to an annual depreciation of 12.0%.

5. c)		
The first method calculates the depreciation for 101 – 12 = 89 months.	1 point	
The value of the car drops by a factor of 0.9894^{89},	1 point	
which is about 0.3873 times the original value.	1 point	*The sales price is about* 1 937 000 *Ft.*
The second method gives an equivalent age of $\frac{91\,250}{15\,000} = \frac{73}{12}$ years, which is about 73 months.	1 point	
The value of the car drops by a factor of 0.988^{73},	1 point	
that is equal to about 0.4142 times the original value.	1 point	*The sales price is about* 2 071 000 *Ft.*
Method 2 is clearly more beneficial for Mr. Kovács.	1 point	
Total:	**7 points**	

6. a)		
One correct example.	2 points	
Determining the mode, median, and mean.	2 points	*Award only 1 point if there is one error, 0 points if there are more.*
Determining that the single mode, the median, and the mean, in this particular order, do form three consecutive terms of an increasing arithmetic progression.	1 point	
Determining the standard deviation.	2 points	
Stating that the standard deviation of the six numbers thrown is not a term of the above arithmetic progression.	1 point	
Total:	**8 points**	

Note: Possible sets of numbers thrown are shown in the table below. (The order of the numbers is arbitrary.)

The numbers	Mode	Median	Mean	*d*	Standard dev.
1, 1, 1, 2, 2, 5	1	1.5	2	0.5	$\sqrt{2}$ (≈ 1.41)
1, 1, 1, 2, 3, 4	1	1.5	2	0.5	$\sqrt{\frac{4}{3}}$ (≈ 1.15)
1, 1, 1, 3, 6, 6	1	2	3	1	$\sqrt{5}$ (≈ 2.24)
1, 2, 2, 3, 4, 6	2	2.5	3	0.5	$\sqrt{\frac{8}{3}}$ (≈ 1.63)
2, 2, 2, 3, 3, 6	2	2.5	3	0.5	$\sqrt{2}$ (≈ 1.41)
2, 2, 2, 3, 4, 5	2	2.5	3	0.5	$\sqrt{\frac{4}{3}}$ (≈ 1.15)
3, 3, 3, 4, 5, 6	3	3.5	4	0.5	$\sqrt{\frac{4}{3}}$ (≈ 1.15)

6. b) Solution 1

(Favourable cases are listed in order of the second number thrown.) If the second number is 1 or 6 then the other two numbers may only be 1 or 6, respectively. This gives 2 cases.	1 point	
If the second number is 2 then the others can be 1 and 3 (two cases), or 2 and 2. This gives 3 cases.	1 point	
If the second number is 3 then the others can be 1 and 5, 2 and 4, or 3 and 3. This gives 5 cases.	1 point	
If the second number is 4 then the others can be 2 and 6, 3 and 5, or 4 and 4. This gives 5 cases.	1 point	
If the second number is 5 then the others can be 4 and 6, or 5 and 5. This gives 3 cases.	1 point	
The number of favourable cases is altogether (2 + 3 + 5 + 5 + 3 =) 18.	1 point	
The total number of cases possible with three dice is (6^3 =) 216.	1 point	
The final probability is $\left(\frac{18}{216}=\right)\frac{1}{12} \approx 0.083$.	1 point	
Total:	**8 points**	

6. b) Solution 2

If the second number thrown is the mean of the other two then the three numbers must form consecutive terms of an arithmetic progression with the second number being the middle term.	1 point	

Consider all favourable cases in order of the common difference d of the progression: $d = 0$, $d = 1$, $d = –1$, $d = 2$, $d = –2$.	1 point	*This point is also due if the correct reasoning is reflected only by the solution.*
$d = 0$ (three identical numbers) gives 6 possible cases.	1 point	
$d = 1$ or $d = –1$ may be possible in four cases each (1-2-3, 2-3-4, 3-4-5, 4-5-6 and the same in reverse order).	1 point	
$d = 2$ or $d = –2$ is possible in two cases each (1-3-5, 2-4-6 and the same in reverse order)	1 point	
The number of favourable cases altogether is $(6 + 4 + 4 + 2 + 2 =)$ 18.	1 point	
The total number of cases possible with three dice is $(6^3 =)$ 216.	1 point	
The final probability is $\left(\frac{18}{216}=\right)\frac{1}{12} \approx 0.083$.	1 point	
Total:	**8 points**	

6. b) Solution 3

If the mean of the first and third numbers thrown is an integer then the sum of these must be even.	1 point	
In each case, the second number thrown may only be one number (their mean). The probability of this is $\frac{1}{6}$.	2 points	
The probability of the sum of the first and third numbers being even is to be determined.	1 point	*This point is also due if the correct reasoning is reflected only by the solution.*
Whether the first number is odd or even, the probability that the third will complete it to an even number is $\frac{1}{2}$.	2 points	*The sum is even if both terms are even or both are odd. The probability of this is* $2\cdot\left(\frac{1}{2}\cdot\frac{1}{2}\right)=\frac{1}{2}$.
(The sum of the first and third numbers being even and the second number being exactly half of their sum are two independent events and therefore) the final probability is the product of the above: $\left(\frac{1}{2}\cdot\frac{1}{6}=\right)\frac{1}{12}$.	2 points	
Total:	**8 points**	

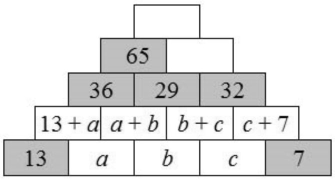
7. a)

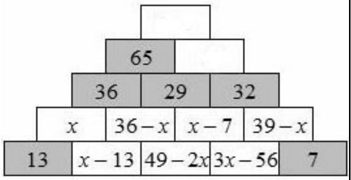
The ratio of the volumes of the circular cone and the cylinder is 1 : 3 and so the percentage of the volume of the cylinder that turns to wood shavings is about 67% (two thirds of the volume).	2 points	

(Use cm and cm^2 for units of measurements.) The volume of the cylinder is $A = 2r^2\pi + 2r\pi m$, from where $10\,000 = 2r^2\pi + 2r\pi \cdot 30$.	1 point	
$\pi r^2 + 30\pi r - 5000 = 0$	1 point	$r^2 + 30r - \frac{5000}{\pi} = 0$
$r_1 \approx 27.62$, $r_2 (\approx -57.62) < 0$	1 point	
(The negative solution is not realistic, so) the base radius of the cylinder is about 27.62 (cm).	1 point	
The volume of the cone is $\left(\frac{30 \cdot 27.62^2 \pi}{3} \approx\right)$ 23 970 cm^3.	1 point	
Total:	**7 points**	

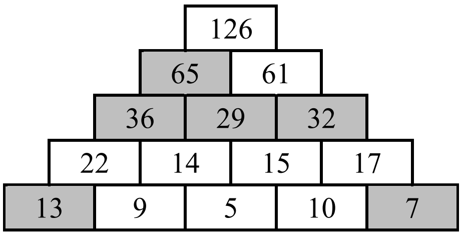
7. b)		
(Use cm and cm^2 for units of measurements.) The surface area of the cylinder is $10\,000 = 2r^2\pi + 2r\pi m$, from where the height of the cylinder is $m = \frac{10\,000 - 2r^2\pi}{2r\pi} = \frac{5000 - r^2\pi}{r\pi}$,	2 points	
the volume of the cylinder is $V = r^2\pi m$, that is $V(r) = 5000r - r^3\pi$ $\left(\text{here } 0 < r < \sqrt{\frac{5000}{\pi}}\right)$.	2 points	
$V'(r) = 5000 - 3r^2\pi$	1 point	
The volume is maximal if $V'(r) = 0$, that is, if $5000 - 3r^2\pi = 0$,	1 point	
from where (as $r > 0$) $r = \sqrt{\frac{5000}{3\pi}} \approx 23.03$ cm (and this is also within the above domain).	1 point	
Here $V''(r) = -6r\pi$, a negative number, therefore it is, in fact, a (global) maximum.	1 point	*At* $r = \sqrt{\frac{5000}{3\pi}}$ *the derivative turns from positive to negative and so the original function must have a (global) maximum here.*
Here $m = \frac{5000 - r^2\pi}{r\pi} \approx 46.07$ cm.	1 point	($m = 2r$)
Total:	**9 points**	

Notes: 1) Award full score if the candidate correctly references that of all closed cylinders of a certain surface area the one with maximal volume is the one that has a square cross-section along the axis, and thereby gives a correct answer.
2) Deduce a total 1 point if the candidate gives any of their answers without a unit of measurement.

8. a) Solution 1		
In the second row from bottom up the numbers are $13 + a$, $a + b$, $b + c$ and $c + 7$.	1 point	
This gives the following equation system: $\left.\begin{aligned}(13+a)+(a+b)&=36\\(a+b)+(b+c)&=29\\(b+c)+(c+7)&=32\end{aligned}\right\}$	2 points	$\left.\begin{aligned}2a+b&=23\\a+2b+c&=29\\b+2c&=25\end{aligned}\right\}$
Express a from the first equation and c from the third: $a=11.5-\frac{b}{2}$ and $c=12.5-\frac{b}{2}$.	2 points	*From the first equation $b = 23 - 2a$. Substitute it into the second:* $c = 3a - 17$.
Write these into the second equation: $\left(11.5-\frac{b}{2}\right)+2b+\left(12.5-\frac{b}{2}\right)=29$,	1 point	*Write these into the third equation:* $(23 - 2a) +$ $+ 2(3a - 17) = 25$.
from where $b = 5$,	1 point	$a = 9$
$a = \left(11.5-\frac{5}{2}=\right)9$ and $c = \left(12.5-\frac{5}{2}=\right)10$.	1 point	$b = (23 - 18 =)\ 5$ *and* $c = (27 - 17 =)\ 10$
Check against the text (e.g. fill in the bottom line of the pyramid with positive integers).	1 point	
Total:	**9 points**	

8. a) Solution 2		
Let x be the number in the first cell of the second row from bottom up. The other numbers in that row are: $36 - x$; $29 - (36 - x) = x - 7$; $32 - (x - 7) = 39 - x$, respectively. (The sum of the two numbers in adjoining cells is the number in the cell above).	2 points	
Similarly, calculate the numbers in the bottom row: $a = x - 13$; $b = (36 - x) - (x - 13) = 49 - 2x$; $c = (x - 7) - (49 - 2x) = 3x - 56$;	2 points	
$(3x - 56) + 7 = 39 - x$,	1 point	
$x = 22$.	1 point	
Substitute x into the appropriate equations. The solution is: $(a; b; c) = (9; 5; 10)$.	2 points	
Check against the text (e.g. fill in the bottom line of the pyramid with positive integers).	1 point	
Total:	**9 points**	

Notes: 1) Award 3 points if the candidate fills in the pyramid (e.g. by trial and error) but does not prove that there are no other possible solutions.
2) The completely filled pyramid is shown:



8. b) **Solution 1**		
The colours for Tolna, Fejér and Somogy are certainly different. Assume, for example, that Tolna is red, Fejér is yellow and Somogy is blue.	2 points	
In this case Bács-Kiskun can only be blue or green, Baranya can only be yellow or green, but they must not be of the same colour.	1 point	*This point is also due if the correct reasoning is reflected only by the solution.*
There are 3 different options to colour Bács-Kiskun and Baranya (in this particular order): blue-yellow, blue-green, green-yellow.	1 point	
The colours of Tolna, Fejér and Somogy may be chosen in $4\cdot 3\cdot 2=24$ different ways.	1 point	
As there are 3 different colouring of Bács-Kiskun and Baranya for each of the above,	1 point	
the total number of different possibilities is $(24\cdot 3=)72$.	1 point	
Total:	**7 points**	

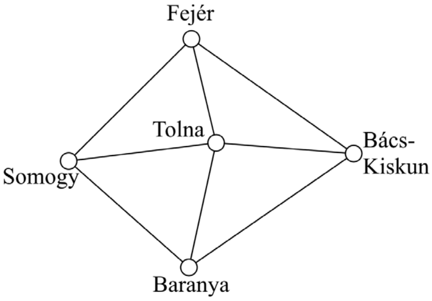
<table>
<tr><td colspan="3">8. b) Solution 2</td></tr>
<tr><td>Let’s suppose that Tolna is red and Fejér is yellow. (Other counties may be coloured in yellow, blue or green following the general rules.)</td><td>1 point</td><td></td></tr>
<tr><td>This gives a total of 6 possible ways of colouring as shown in the table:
<table>
<tr><td>County</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Tolna</td><td>r</td><td>r</td><td>r</td><td>r</td><td>r</td><td>r</td></tr>
<tr><td>Fejér</td><td>y</td><td>y</td><td>y</td><td>y</td><td>y</td><td>y</td></tr>
<tr><td>Somogy</td><td>b</td><td>b</td><td>b</td><td>g</td><td>g</td><td>g</td></tr>
<tr><td>Baranya</td><td>y</td><td>y</td><td>g</td><td>y</td><td>y</td><td>b</td></tr>
<tr><td>Bács-Kiskun</td><td>b</td><td>g</td><td>b</td><td>g</td><td>b</td><td>g</td></tr>
</table></td><td>3 points*</td><td></td></tr>
<tr><td>There are $4 \cdot 3 = 12$ different ways to choose the colour for Tolna and Fejér.</td><td>1 point</td><td></td></tr>
<tr><td>As there are 6 different ways to colour Somogy, Bács-Kiskun and Baranya for each of the options above,</td><td>1 point</td><td></td></tr>
<tr><td>the total number of different ways to colour is therefore $(24 + 48) = 72$.</td><td>1 point</td><td></td></tr>
<tr><td>Total:</td><td>7 points</td><td></td></tr>
</table>

*Note: Deduce 1 point (but no more than a total 3 points altogether) from the 3 points marked with * for every incorrect, missing or duplicate column.*

8. b) **Solution 3**		
Colour the counties of Tolna, Fejér and Somogy (in this particular order): 4 options for Tolna, 3 for Fejér and 2 for Somogy,	1 point	
this is a total of $4 \cdot 3 \cdot 2$ (= 24) options.	1 point	
List the cases with respect to the colour of Bács-Kiskun.	1 point	*This point is also due if the correct reasoning is reflected only by the solution.*
If the colour of Bács-Kiskun is the same as that of Somogy then Baranya may be coloured in 2 different colours. (As it may not be the same as that of Tolna or Somogy.)	1 point	
If the colour of Bács-Kiskun is different from that of Somogy then Bács-Kiskun may only be coloured one way (that is different from the colour of both Tolna, Fejér and Somogy) and Baranya may also be coloured in one way (that is different from its three neighbours).	1 point	
In the first case there are $24 \cdot 2$, while in the second case another $24 \cdot 1$ possible ways to colour,	1 point	
and so the total number of possibilities is $(24 \cdot 2 + 24 \cdot 1 =)\ 72$.	1 point	
Total:	**7 points**	

8. b) **Solution 4**		
List the cases according to the number of colours used. As (for example) Tolna, Somogy and Baranya must all be of different colours, two colours will not be enough.	1 point	*This point is also due if the correct reasoning is reflected only by the solution.*
When 3 colours are used, the county in the middle (Tolna) could be coloured in 4 different ways, there would be 3 options for the (identical) colours of Somogy and Bács-Kiskun, and 2 options for the (also identical) colours of Fejér and Baranya. This is $4 \cdot 3 \cdot 2$ (= 24) different options.	2 points	*There are $4 \cdot 3 \cdot 2$ ways to colour Tolna, Somogy and Baranya, while Fejér and Bács-Kiskun must bear the colour of their respective “opposite” counties. This gives a total 24 possibilities.*
When 4 colours are used there will be 4 different options for the colour of the middle county (Tolna). There are 3 options to choose the colour for two (“opposite”) counties and 2 options to choose which two counties these would be (Fejér and Baranya, or Somogy and Bács-Kiskun). The remaining 2 counties may be coloured in the remaining 2 colours in 2 different ways. This gives $4 \cdot 3 \cdot 2 \cdot 2$ (= 48) options.	3 points	
The total number of options is then (24 + 48 =) 72.	1 point	
Total:	**7 points**	

Note: The colouring of the map may as well be considered as a graph problem. The vertices of the graph are the counties; the edges represent common borderlines.
If the colouring is correct, the two endpoints of each edge are of different colours.



9. a)		
$\frac{1}{n}-\frac{1}{n+2}=\frac{n+2}{n(n+2)}-\frac{n}{n(n+2)}=$	1 point	
$=\frac{2}{n^2+2n}=$	1 point	
$=\frac{2}{(n+1)^2-1}$ (and so the statement is true).	1 point	
Total:	**3 points**	

9. b) Solution 1		
$\frac{2}{2^2-1}+\frac{2}{3^2-1}+\frac{2}{4^2-1}+\frac{2}{5^2-1}=$	1 point	
$=\frac{2}{3}+\frac{2}{8}+\frac{2}{15}+\frac{2}{24}=\frac{80+30+16+10}{120}=\frac{136}{120}=$	1 point	*This point is also due if the candidate obtains the correct answer using a calculator.*
$=\frac{17}{15}$	1 point	
Total:	**3 points**	

9. b) Solution 2		
The sum is written as seen in part a). $\frac{1}{1}-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\frac{1}{4}-\frac{1}{6}=$	1 point	
$=\frac{1}{1}+\frac{1}{2}-\frac{1}{5}-\frac{1}{6}=$	1 point	*This point is also due if the candidate obtains the correct answer using a calculator.*
$=\frac{30+15-6-5}{30}=\frac{17}{15}$	1 point	
Total:	**3 points**	

9. c)		
$S_n = a_1 + a_2 + ... + a_n =$ $= \frac{2}{2^2-1} + \frac{2}{3^2-1} + ... + \frac{2}{(n+1)^2-1}$	1 point	
(As seen in part a)) $S_n = \left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+...+$ $+\left(\frac{1}{n-2}-\frac{1}{n}\right)+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)+\left(\frac{1}{n}-\frac{1}{n+2}\right).$	2 points	
Every fraction, with the exception of four of them, appears with both a positive and a negative sign. The sum of the opposite fractions is zero, only the first two positive and last two negative fractions will not be cancelled: $S_n = \frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$.	2 points	
According to the theorem about the limit of sums and differences,	1 point*	*These points are also due if the correct reasoning is reflected only by the solution.*
as well as $\lim\limits_{n\to\infty}\left(\frac{1}{n+1}\right) = 0$ and $\lim\limits_{n\to\infty}\left(\frac{1}{n+2}\right) = 0$ the limit wanted here is:	2 points*	
$\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty}\left(\frac{3}{2}\right) - \lim\limits_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}\right) =$	1 point*	
$= \frac{3}{2} - (0+0) = \frac{3}{2}$.	1 point*	
Total:	**10 points**	

*Note: The 5 points marked with * may also be given for the following reasoning:*

$S_n = \frac{3n^2+5n}{2n^2+6n+4} =$	1 point	$S_n = \frac{3}{2} - \frac{2n+3}{(n+1)(n+2)} =$
$= \frac{3+\frac{5}{n}}{2+\frac{6}{n}+\frac{4}{n^2}}$	1 point	$= \frac{3}{2} - \frac{\frac{2}{n}+\frac{3}{n^2}}{1+\frac{3}{n}+\frac{2}{n^2}}$
$\lim\limits_{n\to\infty}\left(\frac{1}{n}\right) = 0$, furthermore, according to the theorems about the limits of products, sums and quotients,	1 point	*This point is also due if the correct reasoning is reflected only by the solution.*
the limit wanted here is: $\lim\limits_{n\to\infty} S_n = \frac{3+0}{2+0+0} =$	1 point	$\lim\limits_{n\to\infty} S_n = \frac{3}{2} - \frac{0+0}{1+0+0} =$
$= \frac{3}{2}$.	1 point	